Exercise 13

Solve the Cauchy problem for the linear Klein-Gordon equation

$$u_{tt} - c^2 u_{xx} + a^2 u = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = f(x), \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \text{for } -\infty < x < \infty.$$

Solution

The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t)\, dx,$$

which means the partial derivatives of u with respect to x and t transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n U}{dt^n}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_{tt} - c^2 u_{xx} + a^2 u\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_{tt}\} - c^2 \mathcal{F}\{u_{xx}\} + a^2 \mathcal{F}\{u\} = 0$$

Transform the derivatives with the relations above.

$$\frac{d^2 U}{dt^2} - c^2 (ik)^2 U + a^2 U = 0$$

Expand the second term and factor U.

$$\frac{d^2 U}{dt^2} + (c^2 k^2 + a^2) U = 0 \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. Taking the Fourier transform of the initial conditions gives

$$u(x,0) = f(x) \quad \rightarrow \quad \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$$
$$U(k,0) = F(k) \tag{2}$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \rightarrow \quad \mathcal{F}\left\{\frac{\partial u}{\partial t}(x,0)\right\} = \mathcal{F}\{g(x)\}$$
$$\frac{dU}{dt}(k,0) = G(k). \tag{3}$$

Equation (1) is an ODE in t, so k is treated as a constant. The solution to the ODE is given in terms of sine and cosine.

$$U(k,t) = A(k)\cos\sqrt{c^2k^2+a^2}t + B(k)\sin\sqrt{c^2k^2+a^2}t$$

Apply the first initial condition, equation (2).

$$U(k,0) = A(k) = F(k)$$

In order to apply the second initial condition, differentiate $U(k,t)$ with respect to t.

$$\frac{dU}{dt} = -A(k)\sqrt{c^2k^2+a^2}\sin\sqrt{c^2k^2+a^2}t + B(k)\sqrt{c^2k^2+a^2}\cos\sqrt{c^2k^2+a^2}t$$

Now apply equation (3).

$$\frac{dU}{dt}(k,0) = B(k)\sqrt{c^2k^2+a^2} = G(k) \quad \rightarrow \quad B(k) = \frac{G(k)}{\sqrt{c^2k^2+a^2}}$$

Therefore, the solution to the ODE that satisfies the initial conditions is

$$U(k,t) = F(k)\cos\sqrt{c^2k^2+a^2}t + \frac{G(k)}{\sqrt{c^2k^2+a^2}}\sin\sqrt{c^2k^2+a^2}t.$$

To make $U(k,t)$ easier to work with, introduce a new variable $\omega = \omega(k)$ for the square root term.

$$\omega(k) = \sqrt{c^2k^2+a^2}$$

Then

$$U(k,t) = F(k)\cos\omega t + \frac{G(k)}{\omega}\sin\omega t.$$

In order to change back to $u(x,t)$, we have to take the inverse Fourier transform of $U(k,t)$. It is defined as

$$\mathcal{F}^{-1}\{U(k,t)\} = u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(k,t)e^{ikx}\,dk.$$

Plug $U(k,t)$ into the definition.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[F(k)\cos\omega t + \frac{G(k)}{\omega}\sin\omega t\right]e^{ikx}\,dk$$

Recall that sine and cosine can be written in terms of exponentials using Euler's formula.

$$\cos\omega t = \frac{e^{i\omega t}+e^{-i\omega t}}{2}$$
$$\sin\omega t = \frac{e^{i\omega t}-e^{-i\omega t}}{2i}$$

Substitute these expressions into the equation.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[F(k)\frac{e^{i\omega t}+e^{-i\omega t}}{2} + \frac{G(k)}{\omega}\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right]e^{ikx}\,dk$$

Expand the integrand and factor the terms in $e^{i\omega t}$ and $e^{-i\omega t}$.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \left[\frac{F(k)}{2} + \frac{G(k)}{2i\omega} \right] e^{i\omega t} + \left[\frac{F(k)}{2} - \frac{G(k)}{2i\omega} \right] e^{-i\omega t} \right\} e^{ikx}\, dk$$

Factor the terms in square brackets and distribute e^{ikx}.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{2} \left[F(k) + \frac{G(k)}{i\omega} \right] e^{i(kx+\omega t)} + \frac{1}{2} \left[F(k) - \frac{G(k)}{i\omega} \right] e^{i(kx-\omega t)} \right\} dk$$

Therefore,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(k) e^{i(kx+\omega t)} + B(k) e^{i(kx-\omega t)} \right] dk,$$

where

$$\begin{aligned}
\omega = \omega(k) &= \sqrt{c^2k^2 + a^2} \\
A(k) &= \frac{1}{2} \left[F(k) + \frac{G(k)}{i\omega} \right] \\
B(k) &= \frac{1}{2} \left[F(k) - \frac{G(k)}{i\omega} \right] \\
F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \\
G(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(x)\, dx.
\end{aligned}$$

In the event $a = 0$, then

$$\begin{aligned}
\omega &= ck \\
A(k) &= \frac{1}{2} \left[F(k) + \frac{G(k)}{ick} \right] \\
B(k) &= \frac{1}{2} \left[F(k) - \frac{G(k)}{ick} \right],
\end{aligned}$$

and d'Alembert's solution for the wave equation is obtained as expected (see pg. 37 in the textbook).